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Viscosity methods for zeroes of accretive operators

Paul-Emile Maingé

GRIMMAG, Université des Antilles-Guyane, Département Scientifique Interfacultaire, Campus de Schoelcher, 97230 Cedex, Martinique, FWI

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Abstract

This paper deals with the general iteration method $x_{n+1} := \alpha_n T_n x_n + (1-\alpha_n) J_{r_n}^A x_n$, for calculating a particular zero of A, an m-accretive operator in a Banach space X, T_n being a sequence of nonexpansive self-mappings in X. Under suitable conditions on the parameters and X, we state strong and weak convergence results of (x_n). We also show how to compute a common zero of two m-accretive operators in X.

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1. Introduction

Throughout, X is a real Banach space, A is a (possibly multivalued) m-accretive operator (with domain D_A) in X such that $A^{-1}(0) := \{x \in D_A \,|\, 0 \in Ax\} \neq \emptyset$. We denote by J_r^A (for $r > 0$) the resolvent of A (that is, $J_r^A := (I + rA)^{-1}$) and by $\mathrm{Fix}(T)$ the fixed point set of any operator T in X, that is, $\mathrm{Fix}(T) := \{x \in X,\ x = T(x)\}$; it is well-known that $\mathrm{Fix}(J_r^A) = A^{-1}(0)$. Let (T_n) be a sequence of nonexpansive self-mappings defined on a closed convex set, E, such that $D_A \subset E$.

This paper is concerned with the problem of finding a particular zero of A by using viscosity approximation methods of the form

$$x_{n+1} := \alpha_n T_n x_n + (1-\alpha_n) J_{r_n}^A x_n, \quad \text{with } x_0 \text{ in } E, \tag{1.1}$$

E-mail addresses: Paul-Emile.Mainge@martinique.univ-ag.fr, pemainge@martinique.univ-ag.fr.

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where (α_n), (r_n) are real numbers such that $(\alpha_n) \subset (0, 1)$, $(r_n) \subset (0, \infty)$. More precisely, we study the asymptotic behavior of (1.1) under each of the following conditions on (T_n):

(C1) $T_n := C$ is a contraction on E, namely
$\|Cx - Cy\| \leqslant \varrho \|x - y\|, \quad \forall x, y \in E$, where $\varrho \in (0, 1)$.

(C2) $\bigcap_n \operatorname{Fix}(T_n) \neq \emptyset$ or the sequence (T_n) is bounded on E.

(C3) $\operatorname{Fix}(T_n) = F$ (F being independent of n),
$A^{-1}(0) \cap F \neq \emptyset$.

It is worth recalling that $J_{r_n}^A$ is a nonexpansive mapping from X onto D_A since A is assumed to be m-accretive, so that scheme (1.1) does make sense. In the framework of Hilbert spaces, the two special cases of (1.1) when $T_n := b$ (where b is a fixed element in X) and when $T_n := I$ (identity mapping of X) were investigated by Kamimura and Takahashi [9] for calculating a zero of a maximal monotone operator. In a recent paper, an interesting contribution to both these cases in Banach spaces was due to Dominguez Benavides et al. [6] for approximating a zero of an m-accretive operator. Our aim is to generalize this last work to a more general class of operators T_n.

Note that the proposed method is inspired by Rockafellar's proximal point algorithm [16], Halpern's [8] and Mann's [10] iteration processes. All of these algorithms were first considered in Hilbert spaces and later in Banach spaces (see [4,11,13,14]). It is well-known that proximal algorithm $x_{n+1} := J_{r_n}^A x_n$ converges weakly, but not strongly in general. In [4,11] for instance, additionally to weak convergence results, strong convergence results regarding this proximal iteration are proved for a class of mappings which includes strongly accretive operators (i.e. operators of the form $B + \alpha I$, where B is an m-accretive operator and α a positive real number).

Under suitable conditions on the Banach space X and the parameters (α_n), (r_n), we will prove that (1.1) with condition (C1) always converges strongly to a particular null point of A, while (1.1) with condition (C2) or (C3) converges weakly. As an application of (1.1) with condition (C3), we show how to compute a common zero of two given m-accretive operators in X.

2. Preliminaries

Let $\varphi : [0, \infty) \to [0, \infty)$ be a gauge, namely a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Set

$$\Gamma_\varphi(t) := \int_0^t \varphi(s)\, ds, \quad t \geqslant 0;$$

Γ_φ is obviously a strictly increasing and convex function on $[0, \infty)$. Denote by $J_\varphi : X \to X^*$ the duality map associated with a gauge φ, that is,

$$J_\varphi(x) := \left\{ x^* \in X^* \mid \langle x, x^* \rangle = \|x\| \varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|) \right\}, \quad \forall x \in X. \tag{2.1}$$

The so-called normalized duality map (denoted by J) is the duality map associated with the gauge $\varphi(t) = t$, so that $J_\varphi(x) = \dfrac{\varphi(\|x\|)}{\|x\|} J(x)$ for $x \neq 0$. When J_φ (hence J) is single valued, a main tool of calculus in Banach space is given by the following inequality (see [5]):

$$\Gamma_\varphi(\|x + y\|) \leqslant \Gamma_\varphi(\|x\|) + \langle y, J_\varphi(x + y) \rangle, \quad \forall x, y \in X. \tag{2.2}$$

We also recall that an operator A in X is said to be m-accretive if the following conditions are satisfied:

(i) A is accretive, that is, for all x_1, x_2 in D_A, all $y_1 \in Ax_1$, $y_2 \in Ax_2$, and some $j \in J(x_1 - x_2)$, $\langle y_1 - y_2, j \rangle \geqslant 0$.
(ii) The domain of the resolvent of A is the whole space X.

Besides, a mapping $T : D \to X$ of domain $D \subset X$ is said to be nonexpansive if

$$\|Tx_1 - Tx_2\| \leqslant \|x_1 - x_2\|, \quad \forall x_1, x_2 \in D.$$

To see the connection between accretive operators and nonexpansive mappings, it is worth noting (see [7]) that if T is a nonexpansive mapping on a subset D of X, then $I - T$ is accretive on D. Now, let us recall the main properties of the Banach spaces we use in this paper (for details, we refer the reader to [3,7]):

(1) The norm of X is said to be Fréchet differentiable if

$$\lim_{\lambda \to 0} \frac{1}{\lambda}(\|x + \lambda y\| - \|x\|) \tag{2.3}$$

exists uniformly for $\|y\| = 1$ when x is any fixed element in X. Spaces with a Fréchet differentiable norm include all the classical l^p, L^p spaces $(1 < p < \infty)$.

(2) X is said to be uniformly smooth if the limit (2.3) exists uniformly in the set $\{(x, y) : \|x\| = \|y\| = 1\}$. In such a space, each duality map J_φ is single valued and norm-to-norm uniformly continuous on bounded sets.

(3) X is said to have a weakly continuous duality map J_φ if there exists a gauge φ such that J_φ is single valued and sequentially continuous relative to the weak topologies on both X and X^*, that is, if $(x_n) \subset X$, $x_n \overset{w}{\to} x$, then $J_\varphi(x_n) \overset{w^*}{\to} J_\varphi(x)$. The space l^p $(1 < p < \infty)$ possesses a weakly continuous duality map J_φ with gauge $\varphi(t) = t^{p-1}$.

(4) X is said to be uniformly convex if its modulus of convexity $\delta(\varepsilon)$ is positive for all $\varepsilon \in (0, 2)$, where $\delta(\varepsilon) := \inf\{1 - \frac{1}{2}\|x + y\|;\ \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x - y\| \geqslant \varepsilon\}$.

(5) X satisfying Opial's condition means that if $(x_n) \subset X$ and $x_n \overset{w}{\to} x$, then $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for $y \neq x$. It is well-known that Banach spaces with this property include those which are both uniformly convex and have a weakly continuous duality map.

The following remarks and lemmas are needed in Section 3.

Remark 2.1 (*See Reich [12,14]*). If X is uniformly smooth, then there exists a unique sunny nonexpansive retraction $Q : X \to A^{-1}(0)$ characterized by

$$\langle x - Q(x), J(z - Q(x)) \rangle \leqslant 0, \quad \forall z \in A^{-1}(0), \ \forall x \in X. \tag{2.4}$$

Remark 2.2. If X has a weakly continuous duality map J_φ and if (x_n) is a bounded sequence in X such that $\|x_{n+1} - J^A_{r_n} x_n\| \to 0$ with $r_n \to \infty$, then the set of weak limit points of (x_n) is included in $A^{-1}(0)$. Indeed, let x_{n_k+1} be a subsequence of (x_n) which weakly converges to some $\tilde{x}$ in X and denote by $A_r := \frac{1}{r}(I - J^A_r)$ (for $r > 0$) the so-called Yosida approximation of A. It is immediate that $A_{r_{n_k}}(x_{n_k})$ strongly converges to zero and $J^A_{r_{n_k}}(x_{n_k})$ weakly converges to $\tilde{x}$. By passing to the limit in $A_{r_{n_k}}(x_{n_k}) \in A(J^A_{r_{n_k}}(x_{n_k}))$ and taking into account the fact that the graph of A is weakly–strongly closed, we obtain that $\tilde{x} \in A^{-1}(0)$ (see, for instance, [6,1]).

Remark 2.3. As a classical result, the resolvent identity is written as

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x\right), \quad \forall \lambda > 0, \ \forall \mu > 0, \ \forall x \in X. \tag{2.5}$$

Lemma 2.1. *Let* $(a_n) \subset (0, 1)$ *and* $(b_n) \subset \mathbb{R}$ *and let* $(s_n) \subset [0, \infty)$ *such that*

$$s_{n+1} \leqslant (1 - a_n) s_n + b_n, \quad \forall n \geqslant p \ (\textit{where } p \in \mathbb{N}).$$

The following statements (a) *and* (b) *hold*:

(a) *If* $b_n = \beta a_n$ (*where* β *is a positive constant*), *then*

$$s_{n+1} \leqslant s_p \prod_{k=p}^{n} (1 - a_k) + \beta \left(1 - \prod_{k=p}^{n} (1 - a_k)\right), \quad \forall n \geqslant p.$$

(b) *If* $\limsup_{n\to\infty} \dfrac{b_n}{a_n} \leqslant 0$ *and if* $\sum a_n = \infty$, *then* $\lim_{n\to\infty} s_n = 0$.

Proof. We only indicate the main details of the proof. In case (a), denoting $c_{n,k} = \prod_{j=k}^{n} (1 - a_j)$ for $n \geqslant k$, by a simple induction we get

$$\begin{aligned} s_{n+1} &\leqslant c_{n,p} s_p + \beta \sum_{k=p}^{n-1} a_k c_{n,k+1} + a_n \beta \\ &= c_{n,p} s_p + \beta \left(a_n + \sum_{k=p}^{n-1} (c_{n,k+1} - c_{n,k})\right) = s_p c_{n,p} + \beta (1 - c_{n,p}). \end{aligned}$$

Case (b) is a straightforward consequence of (a) since $b_n \leqslant \varepsilon a_n$ (for any positive ε and large enough n) and noticing that $c_{n,p} \to 0$ (as $n \to \infty$). □

Lemma 2.2 (*See Shioji and Takahashi [17]*). *Let* $c \geqslant 0$ *and let* $(a_0, a_1, \ldots,) \in l^\infty$. *If the following conditions* (i) *and* (ii) *hold*:

(i) $\mu(a_n) \leqslant c$, *for all Banach limit* $\mu(.)$ *on* l^∞,
(ii) $\limsup_{n\to\infty} (a_{n+1} - a_n) \leqslant 0$,

then $\limsup_{n\to\infty} a_n \leqslant c$.

The following lemmas and remarks are useful in Section 4.

Remark 2.4 (*See Xu [18]*). If X is uniformly convex and Ω is a bounded subset of X, then there exists a strictly increasing continuous function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ and such that: $\forall t \in [0, 1]$, $\forall x, y \in \Omega$,

$$\|tx + (1 - t)y\|^2 \leqslant t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t) g(\|x - y\|). \tag{2.6}$$

Remark 2.5. Let W be a subset of X and (x_n) a sequence in X. It is not difficult to see that if X satisfies Opial's condition and if $\lim_{n\to\infty} \|x_n - y\|$ exists for all $y \in W$, then (x_n) has at most one weak limit point in W.

Lemma 2.3 (*See Reich [13]*). *Let Ω be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, and let (U_n) be a sequence of nonexpansive self-mappings on Ω with a nonempty common fixed point set S. If $x_1 \in \Omega$ and $x_{n+1} := U_n x_n$, then $\langle q_1 - q_2, J(f_1 - f_2)\rangle = 0$ for all $f_1, f_2 \in S$ and all q_1, q_2 weak limit points of (x_n).*

3. Strong convergence results

This section is devoted to scheme (1.1) with condition (C1). In the special case $C = b$ (with b in the closure of D_A), Dominguez Benavides et al. [6] obtained the following result: if X is a uniformly smooth Banach space with a weakly continuous duality map J_φ and if the following condition (P0) holds

$$\text{(P0)}: \ \alpha_n \to 0, \quad \sum \alpha_n = \infty, \quad r_n \to \infty,$$

then (x_n) converges strongly to $Q(b)$, where Q is the sunny nonexpansive retraction defined in Remark (2.1). Under the same assumptions, we prove the strong convergence of (1.1) to the unique fixed point of the contraction $Q \circ C$. But also, we cancel the hypothesis of weak continuity of the duality map and we prove that (x_n) given by scheme (1.1)–(C1) still converges strongly to the same limit point with only the following conditions:

$$\text{(P1)}: \ \frac{\alpha_n}{\alpha_{n-1}} \to 1, \ \sum \alpha_n = \infty, \ \frac{1}{\alpha_n}\left(1 - \frac{r_{n-1}}{r_n}\right) \to 0,$$

(P2): $r_n \geqslant \varepsilon$ (for some positive ε).

These conditions are satisfied by the example: $\alpha_n := 1/n$ and $r_n := r \prod_{k=2}^{n} \left(1 + \frac{c}{k \ln k}\right)$ (for any fixed constants $r > 0$ and $c \geqslant 0$), so that $r_n := r$ if $c = 0$, otherwise $r_n \to \infty$.

Theorem 3.1. *Assume X is uniformly smooth and has a weakly continuous duality map J_φ with gauge φ. If* (P0) *is satisfied then (x_n) generated by scheme* (1.1)–(C1) *converges strongly to the unique fixed point of $Q \circ C$, where $Q : X \to A^{-1}(0)$ is the sunny nonexpansive retraction defined in Remark* 2.1.

Proof. It is easily seen that the mapping $Q \circ C$ is a contraction, then it has a unique fixed point (denoted by $\bar{x}$), besides we have

$$x_{n+1} - \bar{x} = \alpha_n (C x_n - C\bar{x}) + (1 - \alpha_n)(J_{r_n}^A x_n - \bar{x}) + \alpha_n (C\bar{x} - \bar{x}). \tag{3.1}$$

By (C1) and since $\bar{x}$ is a fixed point of the nonexpansive mapping $J_{r_n}^A$, we obtain

$$\|x_{n+1} - \bar{x}\| \leqslant (1 - \alpha_n(1 - \varrho))\|x_n - \bar{x}\| + \alpha_n \|C\bar{x} - \bar{x}\|. \tag{3.2}$$

From Lemma (2.1), we deduce the boundedness of the sequence (x_n) if $\prod_{n=0}^{\infty} (1 - (1-\varrho)\alpha_n) = 0$, that is, if $\sum \alpha_n = \infty$. From (2.2) and (3.2), we also have

$$\begin{aligned} &\Gamma_\varphi(\|x_{n+1} - \bar{x}\|) \\ &\quad \leqslant \Gamma_\varphi(\|\alpha_n (C x_n - C\bar{x}) + (1 - \alpha_n)(J_{r_n} x_n - \bar{x})\|) + \alpha_n \left\langle C\bar{x} - \bar{x}, J_\varphi(x_{n+1} - \bar{x})\right\rangle. \end{aligned}$$

As a consequence, Γ_φ being an increasing convex function with $\Gamma_\varphi(0) = 0$, we get

$$\begin{aligned} &\Gamma_\varphi(\|x_{n+1} - \bar{x}\|) \\ &\quad \leqslant (1 - \alpha_n(1 - \varrho))\Gamma_\varphi(\|x_n - \bar{x}\|) + \alpha_n \left\langle C\bar{x} - \bar{x}, J_\varphi(x_{n+1} - \bar{x})\right\rangle. \end{aligned} \tag{3.3}$$

Moreover, it is obviously seen that the sequences (Cx_n) and $(J^A_{r_n}x_n)$ are bounded, so that

$$\|x_{n+1} - J^A_{r_n}x_n\| = \alpha_n\|Cx_n - J^A_{r_n}x_n\| \to 0 \quad \text{if } \alpha_n \to \infty. \tag{3.4}$$

Consequently, by Remark (2.2) it appears that any weak cluster point of (x_n) is in the set $A^{-1}(0)$. Consider a subsequence (x_{n_k}) such that $x_{n_k} \overset{w}{\to} \tilde{x}$ and

$$\limsup_{n\to\infty} \left\langle C\bar{x} - \bar{x}, J_\varphi(x_n - \bar{x})\right\rangle = \lim_{k\to\infty} \left\langle C\bar{x} - \bar{x}, J_\varphi(x_{n_k} - \bar{x})\right\rangle.$$

By Remark 2.1 and by the weak continuity of J_φ, we then obtain

$$\limsup_{n\to\infty} \left\langle C\bar{x} - \bar{x}, J_\varphi(x_n - \bar{x})\right\rangle = \left\langle C\bar{x} - \bar{x}, J_\varphi(\tilde{x} - \bar{x})\right\rangle \leqslant 0. \tag{3.5}$$

Applying Lemma 2.1 to (3.3) and taking into account (3.5), we then have $\lim_{n\to\infty} \Gamma_\varphi(\|x_{n+1} - \bar{x}\|) = 0$, thus $x_n \overset{s}{\to} \bar{x}$, which is the desired result. □

The sequel of our study is inspired by the techniques used by Shioji and Takahashi [17] (see also [8,14]). Let x_t (for $t \in (0,1)$) be the solution of the implicit method $x_t = tx + (1-t)Tx_t$ (where T is a given nonexpansive self-mapping and x a given element in X). It was proved that the iterative process $x_{n+1} := \alpha_n x + (1-\alpha_n)Tx_n$ strongly converges when x_t does so (as $t \to 0$). Following this idea, we define (y_t) as the solution of the implicit method

$$y_t = tCy_t + (1-t)J^A_{r(t)}y_t, \tag{3.6}$$

where $t \in (0,1)$ and $r(.)$ is a real valued function from $(0,1)$ onto $(0,\infty)$. It is obvious that the operator $tC + (1-t)J^A_{r(t)}$ is a contraction on E, so that (y_t) is well defined. In the sequel, we prove the strong convergence of y_t to a zero of A (as $t \to 0$) if $r(t) \geqslant \varepsilon$ (for some positive ε). First of all, we need some preliminaries.

Lemma 3.2. *As $t \to 0$, the solution (y_t) of* (3.6) *has at most one strong limit point in* $A^{-1}(0)$.

Proof. For any q in $A^{-1}(0)$, it is immediate that $|\left\langle J^A_{r(t)}y_t - q, J(y_t - q)\right\rangle| \leqslant \|y_t - q\|^2$ (since $J^A_{r(t)}q = q$). From (3.6), we then obtain

$$\begin{aligned}\|y_t - q\|^2 &= t\langle Cy_t - q, J(y_t - q)\rangle + (1-t)\left\langle J^A_{r(t)}y_t - q, J(y_t - q)\right\rangle \\ &\leqslant t\langle Cy_t - q, J(y_t - q)\rangle + (1-t)\|y_t - q\|^2.\end{aligned}$$

Hence for any q in $A^{-1}(0)$, we have

$$\|y_t - q\|^2 \leqslant \langle Cy_t - q, J(y_t - q)\rangle. \tag{3.7}$$

Considering q_1, q_2 as two strong limit points of (y_t) in $A^{-1}(0)$, we then get

$$\begin{aligned}\|q_1 - q_2\|^2 &\leqslant \langle Cq_1 - q_2, J(q_1 - q_2)\rangle, \\ \|q_2 - q_1\|^2 &\leqslant \langle Cq_2 - q_1, J(q_2 - q_1)\rangle,\end{aligned}$$

so that

$$\begin{aligned}2\|q_1 - q_2\|^2 &\leqslant \langle (Cq_1 - Cq_2) + (q_1 - q_2), J(q_1 - q_2)\rangle \\ &\leqslant (\varrho + 1)\|q_1 - q_2\|^2,\end{aligned}$$

thus $\|q_1 - q_2\| = 0$ since $\varrho \in (0,1)$. □

Lemma 3.3. *Let $\mu(.)$ be a Banach limit on l^∞, $(t_n) \subset (0,1)$, $t_n \to 0$. Set $y_n = y_{t_n}$, where y_t is the solution of* (3.6). *If $r(t_n) \geqslant \varepsilon$* (*for some positive* ε), *then there exists q_ε in $A^{-1}(0)$ such that*

$$\mu(\langle x - q_\varepsilon, J(y_n - q_\varepsilon)\rangle \leqslant 0, \quad \forall x \in X. \tag{3.8}$$

Proof. We use the so-called optimization method (see [15]). Define $f(x) := \mu(\|y_n - x\|^2)$ (for $x \in X$) and $K := \operatorname{argmin}_X f$; K is clearly a nonempty closed convex bounded subset of X. By the resolvent identity (2.5), we have

$$J^A_{r(t_n)} y_n = J^A_\varepsilon \left(\frac{\varepsilon}{r(t_n)} y_n + \left(1 - \frac{\varepsilon}{r(t_n)}\right) J^A_{r(t_n)} y_n \right),$$

hence

$$\|J^A_{r(t_n)} y_n - J^A_\varepsilon y_n\| \leqslant \left|1 - \frac{\varepsilon}{r(t_n)}\right| \times \|y_n - J^A_{r(t_n)} y_n\|. \tag{3.9}$$

This obviously yields

$$\|y_n - J^A_\varepsilon y_n\| \leqslant \left(1 + \left|1 - \frac{\varepsilon}{r(t_n)}\right|\right) \|y_n - J^A_{r(t_n)} y_n\|.$$

Besides, from (3.6), we have $\|y_n - J^A_{r(t_n)} y_n\| \to 0$ as $t_n \to 0$. We therefore obtain

$$\|y_n - J^A_\varepsilon y_n\| \to 0 \quad \text{as } t_n \to 0. \tag{3.10}$$

For any x in K, we deduce that

$$\begin{aligned} f(x) &\leqslant f(J^A_\varepsilon x) = \mu(\|y_n - J^A_\varepsilon x\|^2) \quad \text{(from the definition of } K), \\ &= \mu(\|J^A_\varepsilon y_n - J^A_\varepsilon x\|^2) \quad (\text{since } \|y_n - J^A_\varepsilon y_n\| \to 0), \\ &\leqslant \mu(\|y_n - x\|^2) = f(x) \quad (\text{by nonexpansiveness of } J^A_\varepsilon). \end{aligned}$$

Thus $f(J^A_\varepsilon x) = f(x)$, so that $J^A_\varepsilon(K) \subset K$). Using the fixed point property in smooth Banach spaces, it follows that J^A_ε has a fixed point (denoted by q_ε) in K. For any x in X and $t \in (0,1)$, but also thanks to (2.2), we then get

$$\begin{aligned} 0 &\leqslant \frac{1}{t}(f(q_\varepsilon + t(x - q)) - f(q_\varepsilon)), \\ &= \mu\left(\frac{1}{t}\|(y_n - q_\varepsilon) - t(x - q)\|^2 - \frac{1}{t}\|y_n - q_\varepsilon\|^2\right), \\ &\leqslant 2\mu\left(\langle q_\varepsilon - x, J(y_n - q_\varepsilon - t(x - q_\varepsilon))\rangle\right). \end{aligned}$$

Letting $t \to 0^+$ in this last inequality and using the fact that J is norm-to-norm uniformly continuous on bounded set of X, we obtain the desired result. □

Theorem 3.4. *If $r(t) \geqslant \varepsilon$* (*for some positive* ε), *then the solution y_t of* (3.6) *converges strongly* (*as* $t \to 0$) *to $\bar{x}$, the unique fixed point of the contraction $Q \circ C$, where Q is the nonexpansive sunny retraction defined in Remark* 2.1.

Proof. We use the notations of Lemma 3.3. By (3.7) and Lemma 3.3, we have

$$\begin{aligned}\mu\left(\|y_n-q_\varepsilon\|^2\right) &\leqslant \mu\left(\langle Cy_n-q_\varepsilon, J(y_n-q_\varepsilon)\rangle\right)\\ &= \mu\left(\langle Cy_n-Cq_\varepsilon, J(y_n-q_\varepsilon)\rangle\right)+\mu\left(\langle Cq_\varepsilon-q_\varepsilon, J(y_n-q_\varepsilon)\rangle\right)\\ &\leqslant \mu\left(\langle Cy_n-Cq_\varepsilon, J(y_n-q_\varepsilon)\rangle\right)\\ &\leqslant \varrho\mu\left(\|y_n-q_\varepsilon\|^2\right),\end{aligned}$$

thus $\mu\left(\|y_n-q_\varepsilon\|^2\right)=0$ (since $\varrho\in(0,1)$). Therefore, there exists a subsequence of (y_n) that converges strongly to q_ε. By (3.10) it is easily seen that any strong limit point of (y_t) (as $t\to 0$) is in $A^{-1}(0)$, while Lemma 3.2 gives uniqueness of such a limit point. We deduce that $y_t \overset{s}{\to} q_\varepsilon$ (when $t\to 0$). It remains to identify the limit q_ε. Using (3.6) and for any x in $A^{-1}(0)$, we have $y_t-Cy_t=-(1/t-1)(I-J^A_{r(t)})y_t$ and $(I-J^A_{r(t)})x=0$, so that

$$\begin{aligned}\langle y_t-Cy_t, J(y_t-x)\rangle &= -(1/t-1)\left\langle (I-J^A_{r(t)})y_t-(I-J^A_{r(t)})x, J(y_t-x)\right\rangle\\ &\leqslant 0,\end{aligned}$$

because $I-J^A_{r(t)}$ is accretive. As $t\to 0$ in this last inequality, it follows that

$$\langle q_\varepsilon-Cq_\varepsilon, J(q_\varepsilon-x)\rangle \leqslant 0, \quad \forall x\in A^{-1}(0).$$

By Remark 2.1, we then obtain $q_\varepsilon=Q(Cq_\varepsilon)$, hence $q_\varepsilon=\bar{x}$, which ends the proof. □

Before stating our convergence result about the sequence (x_n), we also need the following:

Lemma 3.5. *If* (P1) *holds, then* (x_n) *given by scheme* (1.1)–(C1) *satisfies*

$$\|x_{n+1}-x_n\|\to 0. \tag{3.11}$$

Proof. For simplicity's sake, we write J_n instead of $J^A_{r_n}$; hence by relation (1.1) we have

$$\begin{aligned}x_{n+1}-x_n = &(1-\alpha_n)(J_nx_n-J_{n-1}x_{n-1})+\alpha_n(Cx_n-Cx_{n-1})\\ &+(\alpha_n-\alpha_{n-1})(Cx_{n-1}-J_{n-1}x_{n-1}),\end{aligned}$$

so that

$$\begin{aligned}\|x_{n+1}-x_n\| &\leqslant (1-\alpha_n)\|J_nx_n-J_{n-1}x_{n-1}\|+\alpha_n\varrho\|x_n-x_{n-1}\|\\ &\quad +|\alpha_n-\alpha_{n-1}|\times\|Cx_{n-1}-J_{n-1}x_{n-1}\|\\ &\leqslant (1-\alpha_n(1-\varrho))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\times\|Cx_{n-1}-J_{n-1}x_{n-1}\|\\ &\quad +(1-\alpha_n)\|J_nx_{n-1}-J_{n-1}x_{n-1}\|.\end{aligned}$$

Using (2.5), we also have

$$\begin{aligned}\|J_nx_{n-1}-J_{n-1}x_{n-1}\| &= \left\|J_{n-1}\left(\frac{r_{n-1}}{r_n}x_{n-1}+\left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_{n-1}\right)-J_{n-1}x_{n-1}\right\|\\ &\leqslant \left|1-\frac{r_{n-1}}{r_n}\right|\times\|x_{n-1}-J_nx_{n-1}\|.\end{aligned}$$

Combining the last two inequalities yields

$$\begin{aligned}\|x_{n+1}-x_n\| \leqslant &(1-\alpha_n(1-\varrho))\|x_n-x_{n-1}\|+M_1|\alpha_n-\alpha_{n-1}|\\ &+M_2\left|1-\frac{r_{n-1}}{r_n}\right|,\end{aligned}$$

where M_1, M_2 are positive constants (independent of n because of the boundedness of (x_n)). From Lemma 2.1, the desired result follows. □

At once, we prove the strong convergence of (x_n) and (y_t) to the same limit.

Theorem 3.6. *If X is uniformly smooth and if* (P1) *and* (P2) *hold, then (x_n) generated by scheme* (1.1)–(C1) *converges strongly to $\bar{x}$, the fixed point of the contraction $Q \circ C$, where Q is the nonexpansive sunny retraction defined in Remark* 2.1.

Proof. With similar arguments as in the proof of Theorem 3.1, we obtain the boundedness of (x_n), but also

$$\|x_{n+1} - \bar{x}\|^2 \leqslant (1 - \alpha_n(1 - \varrho))\|x_n - \bar{x}\|^2 + \alpha_n \langle C\bar{x} - \bar{x}, J(x_{n+1} - \bar{x})\rangle . \tag{3.12}$$

Consider μ a Banach limit and let y_t be the solution of (3.6) in the particular case $r(t) := \varepsilon$. We have $\|x_{n+1} - x_n\| \to 0$ by Lemma 3.5, while $\|x_{n+1} - J_{r_n}^A x_n\| \to 0$ (as $\alpha_n \to 0$). It follows that $\|x_n - J_{r_n}^A x_n\| \to 0$. In the same manner as (3.9) was established, we get $\|x_n - J_\varepsilon^A x_n\| \to 0$ since $\|J_{r_n}^A x_n - J_\varepsilon^A x_n\| \to 0$ (for $r_n \geqslant \varepsilon$). It follows that

$$\begin{aligned}\mu\left(\|x_n - J_\varepsilon^A y_t\|^2\right) &= \mu\left(\|J_\varepsilon^A x_n - J_\varepsilon^A y_t\|^2\right)\\ &\leqslant \mu\left(\|x_n - y_t\|^2\right).\end{aligned} \tag{3.13}$$

On the other hand, by (3.6) we have

$$x_n - y_t = (1-t)(x_n - J_\varepsilon^A y_t) + t(x_n - Cy_t),$$

so that

$$\|x_n - y_t\|^2 \leqslant (1-t)^2\|x_n - J_\varepsilon^A y_t\|^2 + 2t\,\langle x_n - Cy_t, J(x_n - y_t)\rangle ,$$

hence

$$(1-2t)\|x_n - y_t\|^2 \leqslant (1-t)^2\|x_n - J_\varepsilon^A y_t\|^2 + 2t\,\langle y_t - Cy_t, J(x_n - y_t)\rangle .$$

Combining this result with (3.13) leads to

$$(1-2t)\mu\left(\|x_n - y_t\|^2\right) \leqslant (1-t)^2\mu\left(\|x_n - y_t\|^2\right) + 2t\mu\left(\langle y_t - Cy_t, J(x_n - y_t)\rangle\right),$$

that is,

$$\mu\left(\langle Cy_t - y_t, J(x_n - y_t)\rangle\right) \leqslant t\mu\left(\|x_n - y_t\|^2\right).$$

Consequently, since X is uniformly smooth and $y_t \xrightarrow{s} \bar{x}$ by Theorem 3.4, passing to the limit in this last inequality yields

$$\mu\left(\langle C\bar{x} - \bar{x}, J(x_n - \bar{x})\rangle\right) \leqslant 0.$$

Moreover, since $\|x_{n+1} - x_n\| \to 0$ by Lemma 3.5, we obviously get

$$\lim_{n\to\infty} |\langle C\bar{x} - \bar{x}, J(x_{n+1} - \bar{x})\rangle - \langle C\bar{x} - \bar{x}, J(x_n - \bar{x})\rangle| = 0,$$

because J is norm-to-norm continuous. Hence by Lemma 2.2, we deduce that

$$\limsup_{n\to\infty} \langle C\bar{x} - \bar{x}, J(x_n - \bar{x})\rangle \leqslant 0. \tag{3.14}$$

By Lemma 2.1, thanks to (3.12) and (3.14), we conclude to $x_n \overset{s}{\to} \bar{x}$, which completes the proof. □

4. Weak convergence results

This section is concerned with scheme (1.1) under condition (C2) or (C3). Concerning the particular case $T_n = I$, which was treated in [6], the following two results are proved:

- If X is uniformly convex with both a Fréchet differentiable norm and a weakly continuous duality map J_φ and if $\alpha_n \to 0$ and $r_n \to \infty$, then (x_n) weakly converges to a point in $A^{-1}(0)$.
- If X is a uniformly convex space either with a Fréchet differentiable norm or which satisfies Opial's condition and if $(\alpha_n) \subset [\varepsilon, 1-\varepsilon]$ and $r_n \geqslant \varepsilon$ (for some $\varepsilon > 0$), then (x_n) weakly converges to a point in $A^{-1}(0)$.

We present here some complementary results and we adapt the iteration method for calculating a common zero of two m-accretive operators in X.

Theorem 4.1. *Suppose X has a weakly continuous duality map J_φ and satisfies Opial's condition. If the following conditions hold*

$$\text{(i)} \sum \alpha_n < \infty \quad \text{(ii)}\ r_n \to \infty,$$

then (x_n) given by scheme (1.1)–(C2) (*respectively* (1.1)–(C3)) *converges weakly to a point in* $A^{-1}(0)$.

Proof. Taking any $\tilde{x}$ in $A^{-1}(0)$, we have

$$x_{n+1} - \tilde{x} = \alpha_n(T_n x_n - T_n\tilde{x}) + (1-\alpha_n)(J^A_{r_n}x_n - \tilde{x}) + \alpha_n(T_n\tilde{x} - \tilde{x}), \tag{4.1}$$

so that

$$\|x_{n+1} - \tilde{x}\| \leqslant \|x_n - \tilde{x}\| + \alpha_n\|T_n\tilde{x} - \tilde{x}\|. \tag{4.2}$$

Under condition (C2) or (C3), the quantity $\|T_n\tilde{x} - \tilde{x}\|$ is bounded. This is obvious for (C2). To see this for (C3), take $z \in \bigcap_n \mathrm{Fix}(T_n)$, so that $\|T_n\tilde{x} - \tilde{x}\| \leqslant \|\bar{x} - z\| + \|z - \tilde{x}\|$. As a consequence, by (4.2), there exists $M > 0$ such that $\|x_{n+1} - \tilde{x}\| \leqslant \|x_n - \tilde{x}\| + \alpha_n M$ $(n \geqslant 0)$. Noting that $\alpha_n = \beta_n - \beta_{n+1}$, where $\beta_j = \sum_{k\geqslant 0}\alpha_k - \sum_{k=0}^{j-1}\alpha_k \to 0$ (as $j \to \infty$), it follows that the sequence $(\beta_n M + \|x_n - \tilde{x}\|)$ is decreasing, then it converges and so does $\|x_n - \tilde{x}\|$. By Remark 2.5 and Opial's condition, we conclude that (x_n) has at most one weak cluster point in $A^{-1}(0)$. Moreover, it is immediate that the sequences (x_n), $(J^A_{r_n}x_n)$ and (Cx_n) are bounded. Using the definition of the scheme, we therefore have $\|x_{n+1} - J^A_{r_n}x_n\| \to 0$. By Remark 2.2, we then deduce that any weak limit point of (x_n) is in $A^{-1}(0)$, which leads to the desired result. □

Lemma 4.2. *Assume X is uniformly convex and let (U), (V) be two nonexpansive operators in X such that* $\mathrm{Fix}(U) \cap \mathrm{Fix}(V) \neq \emptyset$. *If* $\alpha \in (0,1)$, *then* $\mathrm{Fix}(T) = \mathrm{Fix}(U) \cap \mathrm{Fix}(V)$, *where* $T := \alpha U + (1-\alpha)V$.

Proof. Let x be a fixed point of T and $\tilde{x}$ in $\mathrm{Fix}(U) \cap \mathrm{Fix}(V)$. We then have

$$x - \tilde{x} = \alpha(Ux - \tilde{x}) + (1-\alpha)(Vx - \tilde{x}). \tag{4.3}$$

By Remark 2.4, there exists a strictly increasing and continuous function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, such that

$$\|x - \tilde{x}\|^2 \leqslant \alpha\|Ux - \tilde{x}\|^2 + (1-\alpha)\|Vx - \tilde{x}\|^2 - \alpha(1-\alpha)g(\|Ux - Vx\|).$$

By nonexpansiveness of U and V, we obtain

$$\|x - \tilde{x}\|^2 \leqslant \|x - \tilde{x}\|^2 - \alpha(1-\alpha)g(\|Ux - Vx\|).$$

Thus $g(\|Ux - Vx\|) \leqslant 0$, that is, $Ux = Vx$, hence $x = Ux = Vx$, so that $x \in \mathrm{Fix}(U) \cap \mathrm{Fix}(V)$. Conversely, it is obvious that any $x \in \mathrm{Fix}(U) \cap \mathrm{Fix}(V)$ is a fixed point of T, which ends the proof. □

Lemma 4.3. *Assume X is uniformly convex. If $(\alpha_n) \subset [\varepsilon, 1-\varepsilon]$ (for some $\varepsilon > 0$), then (x_n) given by scheme* (1.1)–(C3) *satisfies*

$$\lim_{n\to\infty} \|x_{n+1} - T_n x_n\| = \lim_{n\to\infty} \|x_{n+1} - J_{r_n}^A x_n\| = \lim_{n\to\infty} \|T_n x_n - J_{r_n}^A x_n\| = 0. \tag{4.4}$$

Proof. Considering $\tilde{x}$ in $\mathrm{Fix}(T_n) \cap A^{-1}(0)$, by Lemma 4.2 we get

$$x_{n+1} - \tilde{x} = \alpha_n(T_n x_n - \tilde{x}) + (1-\alpha_n)(J_{r_n}^A x_n - \tilde{x}). \tag{4.5}$$

By the uniform convexity of X, there exists a strictly increasing and continuous function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, such that

$$\|x_{n+1} - \tilde{x}\|^2 \leqslant \alpha_n\|T_n x_n - \tilde{x}\|^2 + (1-\alpha_n)\|J_{r_n}^A x_n - \tilde{x}\|^2 - \alpha_n(1-\alpha_n)g(\|T_n x_n - J_{r_n}^A x_n\|).$$

Using the nonexpansiveness of T_n, we get

$$\alpha_n(1-\alpha_n)g(\|T_n x_n - J_{r_n}^A x_n\|) \leqslant \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2.$$

It follows that $\sum_{n\geqslant 0} g(\|T_n x_n - J_{r_n}^A x_n\|) < \infty$ if $(\alpha_n) \subset [\varepsilon, 1-\varepsilon]$, so that $\|T_n x_n - J_{r_n}^A x_n\| \to 0$. By definition of the scheme, we then obtain the desired result. □

Theorem 4.4. *Assume X is uniformly convex. If $(\alpha_n) \subset [\varepsilon, 1-\varepsilon]$ (for some $\varepsilon > 0$) and $r_n \to \infty$, then any weak cluster point of (x_n) given by scheme* (1.1)–(C3) *is in $A^{-1}(0)$. If X has also a weakly continuous duality map J_φ, then (x_n) converges weakly to a point in $A^{-1}(0)$.*

Proof. Let (x_{n_k}) be a converging subsequence of (x_n) such that $x_{n_k} \overset{w}{\to} \tilde{x}$. By definition of the scheme, we have

$$x_{n+1} = T_n x_n + (1-\alpha_n)(J_{r_n}^A x_n - T_n x_n).$$

Hence from Lemma 4.3, we obtain

$$T_{n_k-1} x_{n_k-1} \overset{w}{\to} \tilde{x}, \tag{4.6}$$

since $\|J_{r_n}^A x_n - T_n x_n\| \to 0$. Moreover, for any fixed $\lambda > 0$ and by the resolvent identity, we have

$$J_{r_n}^A x_n = J_\lambda^A \left(\frac{\lambda}{r_n} x_n + \left(1 - \frac{\lambda}{r_n}\right) J_{r_n}^A x_n \right),$$

so that

$$\begin{aligned}\|J_{r_n}^A x_n - J_\lambda^A \circ T_n x_n\| &\leqslant \left\| J_{r_n}^A x_n - T_n x_n + \frac{\lambda}{r_n}(x_n - J_{r_n}^A x_n) \right\| \\ &\leqslant \|J_{r_n}^A x_n - T_n x_n\| + \frac{\lambda}{r_n} \|x_n - J_{r_n}^A x_n\|. \end{aligned} \tag{4.7}$$

From Lemma 4.3, as (x_n), $(J_{r_n}^A x_n)$ are bounded sequences and $r_n \to \infty$, we get $\|T_n x_n - J_\lambda^A \circ T_n x_n\| \to 0$, so that

$$\left(I - J_\lambda^A\right) T_{n_k - 1} x_{n_k - 1} \overset{s}{\to} 0. \tag{4.8}$$

By combining this result with (4.6) and since the operator $I - J_\lambda^A$ is demiclosed, we deduce that $\tilde{x} \in \mathrm{Fix}(J_\lambda^A) = A^{-1}(0)$. The weak convergence of (x_n) is due to the fact that the uniformly convex Banach space X satisfies Opial's condition if it has a weakly continuous duality map, which ensures the uniqueness of a weak limit point. □

Theorem 4.5. *Suppose X is uniformly convex and has either a Fréchet differentiable norm or a weakly continuous duality map J_φ. Let B be an m-accretive operator in X such that $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Assume the following conditions*:

$$(\alpha_n) \subset [\varepsilon, 1 - \varepsilon] \quad (\textit{for some } \varepsilon > 0), \quad r_n \to \infty.$$

Then (x_n) generated by scheme (1.1)–(C3), *with $T_n = J_{r_n}^B$, converges weakly to a point in $A^{-1}(0) \cap B^{-1}(0)$.*

Proof. By Theorem 4.4, it is easily seen that $w_w(x_n)$ (the set of weak limit points of (x_n)) is included in $A^{-1}(0) \cap B^{-1}(0)$. When X has a Fréchet differentiable norm, we set $U_n := \alpha_n J_{r_n}^B + (1 - \alpha_n) J_{r_n}^A$, hence scheme (1.1) may be rewritten as $x_{n+1} := U_n x_n$. From Lemma 4.2, we also have $\mathrm{Fix}(U_n) = A^{-1}(0) \cap B^{-1}(0)$, so that $\langle q_1 - q_2, J(f_1 - f_2)\rangle = 0$ for all q_1, q_2 in $w_w(x_n)$ and all f_1, f_2 in $\mathrm{Fix}(T_n)$. It follows that (x_n) has exactly one weak limit point which belongs to $A^{-1}(0) \cap B^{-1}(0)$, hence (x_n) weakly converges. When X has a weakly continuous duality map, Theorem 4.4 gives the weak convergence of (x_n). □

Remark 4.1. For any given maximal monotone operators A, B in a real Hilbert space H and a positive real number λ, it is proved in [2] that the alternating resolvent method

$$x_{n+1} := (J_\lambda^A \circ J_\lambda^B) x_n$$

converges weakly to an element of the set of solution (assumed to be nonempty) of inclusion problem

$$\text{find } x \in H \text{ such that } 0 \in Ax + B_\lambda x, \tag{4.9}$$

where B_λ is the Yosida approximation of B (that is, $B_\lambda := \frac{1}{\lambda}(I - J_\lambda^B)$). Moreover, it is immediate that any point in $A^{-1}(0) \cap B^{-1}(0)$ (provided that it is a nonempty set) is a solution of (4.9).

Consequently, Theorem 4.5 provides an alternative iterative method for approximating a solution of (4.9).

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